This situation is encountered in cases when, connected to the output terminals of the amplifier, is an arbitrary active circuit (with a voltage or a current source) or a circuit with storage properties which, by the principle of compensation, can be replaced by such an active circuit. For analysis, we can consider that connected to the output terminals of the inverting non-linear amplifier are two active loads (Fig. 227a) in which the effect of the output resistances $R_{oP}$ and $R_{oN}$ of the amplifiers $B_P$ and $B_N$ in the cut-off state has been included.

To determine the behaviour of the circuit of Fig. 227a, we shall first assume that the two feedback loops are interrupted at points marked X, and that the voltage $v_i \approx 0$. All the circuit quantities will be denoted for this case by the additional subscript X. On the above assumptions, the voltages across the output terminals $2-2'$ and $3-3'$ will be

$$v_{2X} = \frac{R_P}{R_P + R_{i2}} v_{i2}, \qquad v_{3X} = \frac{R_N}{R_N + R_{i3}} v_{i3}. \tag{10.24}$$

If $v_{2X} \geqq v_{3X}$, we can determine which of the two amplifiers $B_P$ and $B_N$ is in the active mode on the basis of this consideration. Let us determine the (fictitious) current

$$i_{iX} = i_1 + i_{PX} + i_{NX} \approx \frac{v_1}{R_A} + \frac{v_{i2}}{R_{i2} + R_P} + \frac{v_{i3}}{R_{i3} + R_N}, \tag{10.25}$$

which in the given conditions would flow to the input terminal of the amplifier OA. From its polarity, we can determine the polarity of the output voltage of the amplifier OA and, consequently, which of the amplifiers $B_P$ and $B_N$ will amplify. From Eq. (10.25) we derive that

$$i_{1X} \gtreqless 0 \quad \text{for} \quad v_1 \gtreqless V_r, \tag{10.26}$$

where

$$V_r = -R_A\left(\frac{v_{i2}}{R_{i2} + R_P} + \frac{v_{i3}}{R_{i3} + R_N}\right). \tag{10.27}$$

If we now connect the two feedback loops to the non-linear operational amplifier, then for $v_1 > V_r$ the amplifier $B_N$ will operate while the amplifier $B_P$ will be cut off; conversely, for $v_1 < V_r$ the amplifier $B_P$ will be in operation and the amplifier $B_N$ will not amplify. In the special case, when $v_1 = V_r$, both amplifiers are cut off and the output voltages $v_2 = v_{2X}$ and $v_3 = v_{3X}$. The behaviour of the inverting non-linear amplifier with active loads, for which $v_{2X} \geqq v_{3X}$, can be described by the transfer characteristics



$$v_2 = \begin{cases} v_{2X} & \text{for } v_1 \geqq V_r, \\ -\dfrac{R_P}{R_A} v_1 - \dfrac{R_P}{R_N} v_{3X} & \text{for } v_1 < V_r; \end{cases} \tag{10.28}$$

$$v_3 = \begin{cases} -\dfrac{R_N}{R_A} v_1 - \dfrac{R_N}{R_P} v_{2X} & \text{for } v_1 > V_r, \\ v_{3X} & \text{for } v_1 \leqq V_r. \end{cases} \tag{10.29}$$

An example of the transfer characteristics is given in Fig. 227b.

The situation is different in the case which is characterized by the condition $v_{2X} < v_{3X}$, where the voltages $v_{2X}$ and $v_{3X}$ are given by Eq. (10.24). In a circuit dimensioned like this it occurs that over the interval characterized on the output side by the condition $v_2, v_3 \in (v_{2X}, v_{3X})$ the two amplifiers $B_P$ and $B_N$ are always operative at the same time. This disturbs the correct operation of the linear feedback in this region, and the output characteristics exhibit a curved knee in this region, as can be seen from Fig. 227c. The intersections of the two characteristics lie on the straight line $v_{2,3} = v_1 R_2 R_3/(R_1 R_2 + R_1 R_3)$.

## 10.2 RESONANCE PHENOMENA IN NON-LINEAR AND PARAMETRIC CIRCUITS

### 10.2.1 Resonance in non-linear resonant circuits

The term non-linear resonant circuit is applied to denote a second-order circuit in which *at least one storage element is non-linear*. In recent years, semiconductor and ferrite elements have come to be employed in such circuits. The capacitance of PN junctions of semiconductor elements depends on the junction voltage, while the inductance of coils with ferrite cores depends on the current flowing; they are evidently non-linear elements.

Depending on the nature of the applied non-linear storage element, we distinguish non-linear resonant circuits with non-linear capacitor and with non-linear inductor. Depending on the connection of the elements, we distinguish series and parallel resonant circuits [9], [33], [45].

A change in the capacitance or inductance of the element together with changes in the voltage and current gives rise to phenomena in the circuit

which differ qualitatively from the phenomena that can be observed in resonant circuits with constant parameters $L$ and $C$. To investigate the operation of non-linear resonant circuits, we usually apply the method of equivalent linearization or the method of slowly changing amplitudes. If the quality factor of the resonant circuit is high enough, the circuit is highly selective, and when a harmonic voltage acts on such a circuit, we are justified in assuming that the response will also be almost harmonic.

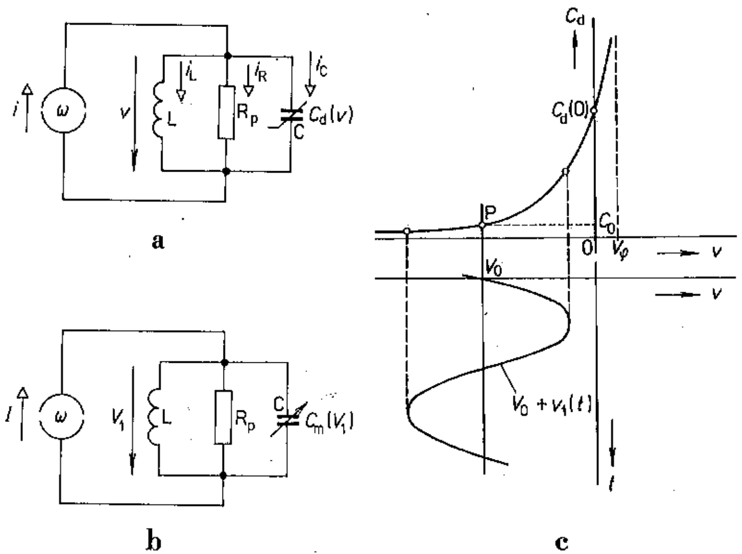


Fig. 228. a) A parallel resonant circuit with a non-linear capacitor, b) its model for the first harmonic components of current and voltage, c) the farad-volt characteristic of the capacitive diode

Resonance in the parallel resonant circuit with a non-linear capacitor. We shall find the resonance characteristics of the parallel resonant circuit of Fig. 228. In the analysis, we shall use the method of equivalent linearization, which was discussed in Section 5.2.1. In accordance with experimental findings, we shall assume that the resonant circuit has marked selective properties so that the higher harmonic components of the current produce a negligible voltage across the resonant circuit. Across the circuit, only the current of frequency $\omega$ will give rise to the harmonic voltage

$$v \approx v_1 = V_1 \sin(\omega t). \tag{10.30}$$



Capacitive diodes are now the most frequently used form of non-linear capacitors. We shall consider a capacitive diode with an abrupt PN junction, whose characteristics can be approximated by the function

$$C_d(v) = \frac{C_d(0)}{\sqrt{1 - v/V_\varphi}}. \tag{10.31}$$

Here, $v$ is the voltage applied to the diode, $C_d(0)$ is the differential capacitance at $v = 0$ (see Fig. 228c), and $V_\varphi$ is the contact potential of the junction. If $v = V_0 + v_1$, where $V_0$ is the bias determining the quiescent operating point P, Eq. (10.31) can be rewritten in the form

$$C_d(v_1) = C_0(1 + \sqrt{a}v_1)^{-1/2}, \tag{10.32}$$

where

$$C_0 = C_d(0)\sqrt{\frac{V_\varphi}{V_\varphi - V_0}}, \qquad \sqrt{a} = \frac{1}{V_0 - V_\varphi}. \tag{10.33}$$

We usually work with $V_1 < V_\varphi - V_0$, hence $|\sqrt{a}V_1| < 1$. Then expression (10.32) can be expanded into a power series, and we shall concentrate on its first three terms only. For the capacitive diode under consideration we thus obtain

$$C_d(v_1) \approx C_0\left(1 - \frac{1}{2}\sqrt{a}v_1 + \frac{3}{8}av_1^2\right) = C_0 + a_1 v_1 + a_2 v_1^2, \tag{10.34}$$

where $a_1 = C_0\sqrt{a}/2$, $a_2 = 3C_0 a/8$. The current flowing through the capacitor is

$$\begin{aligned} i_C &= \frac{dq}{dt} = C_d(v_1)\frac{dv_1}{dt} = C_d(v_1)\,\omega V_1 \cos(\omega t) \\ &= \left(C_0 + \frac{a_2}{2}V_1^2 + a_1 V_1 \sin(\omega t) - \frac{a_2}{2}V_1^2 \cos(2\omega t)\right)\omega V_1 \cos(\omega t). \end{aligned} \tag{10.35}$$

The first harmonic component of this current is

$$i_{C1} = \left(C_0 + \frac{a_2}{2}V_1^2\right)\omega V_1 \cos(\omega t), \tag{10.36}$$

so that the equivalent capacitance is

$$C_e(V_1) = C_0 + \frac{a_2}{2}V_1^2 = C_0\left(1 + \frac{3}{16}aV_1^2\right). \tag{10.37}$$

Increasing the $C_e$ with increasing amplitude $V_1$ leads to a reduction in the resonant frequency of the resonant circuit. When calculating the resonant

frequency, we can start from the equality of the inductive and the capacitive reactance

$$\omega_r L = \frac{1}{\omega_r C_e(V_1)} = \frac{1}{\omega_r C_0\left(1 + \frac{3}{16} aV_1^2\right)}.$$

Denoting $\omega_0 = 1/\sqrt{LC_0}$, we obtain the result in the form

$$\omega_r(V_1) = \frac{\omega_0}{\sqrt{1 + \frac{3}{16} aV_1^2}}. \tag{10.38}$$

The dash-and-dot curve of Fig. 229a is inverse to the curve $\omega_r(V_1)/\omega_0$, which gives the dependence of the relative resonant frequency $\omega_r/\omega_0$ on the amplitude $V_1$ of the voltage across the capacitor.

Now we shall write the equation for the non-linear circuit, and by solving it we shall obtain the module and the argument (phase) characteristics. Following from Fig. 228b, the complex amplitude of the voltage $\mathbf{V}_1$ will be expressed by the relation

$$\mathbf{V}_1 = \frac{\mathbf{I}}{\frac{1}{R_p} + \frac{1}{j\omega L} + j\omega C_e(V_1)} = \frac{j\omega L R_p I}{R_p[1 - \omega^2 LC_e(V_1)] + j\omega L}. \tag{10.39}$$

The module of this complex amplitude is

$$V_1 = \sqrt{\frac{I^2\omega^2 L^2 R_p^2}{R_p^2[1 - \omega^2 LC_e(V_1)]^2 + \omega^2 L^2}}. \tag{10.40}$$

This equation can be rewritten in the form

$$[1 - \omega^2 LC_e(V_1)]^2 + \frac{1}{Q^2}\left[1 - \left(\frac{V_{max}}{V_1}\right)^2\right] = 0, \tag{10.41}$$

where $V_{max} = IR_p$ and $Q = R_p/(\omega_0 L) \approx R_p/(\omega L)$ is the quality factor of the resonant circuit when $V_1 \to 0$. After mathematical rearrangement, we obtain the biquadratic equation

$$\omega^4 - \frac{2}{LC_e(V_1)}\omega^2 + \frac{1}{Q^2 L^2 C_e^2(V_1)}\left[1 + Q^2 - \left(\frac{V_{max}}{V}\right)^2\right] = 0, \tag{10.42}$$

with the solution

$$\omega^2 = \frac{1}{LC_e(V_1)}\left\{1 \pm \sqrt{1 - \left[1 - \frac{1}{Q^2}\left(\frac{V_{max}^2}{V_1^2} - 1\right)\right]}\right\}. \tag{10.43}$$



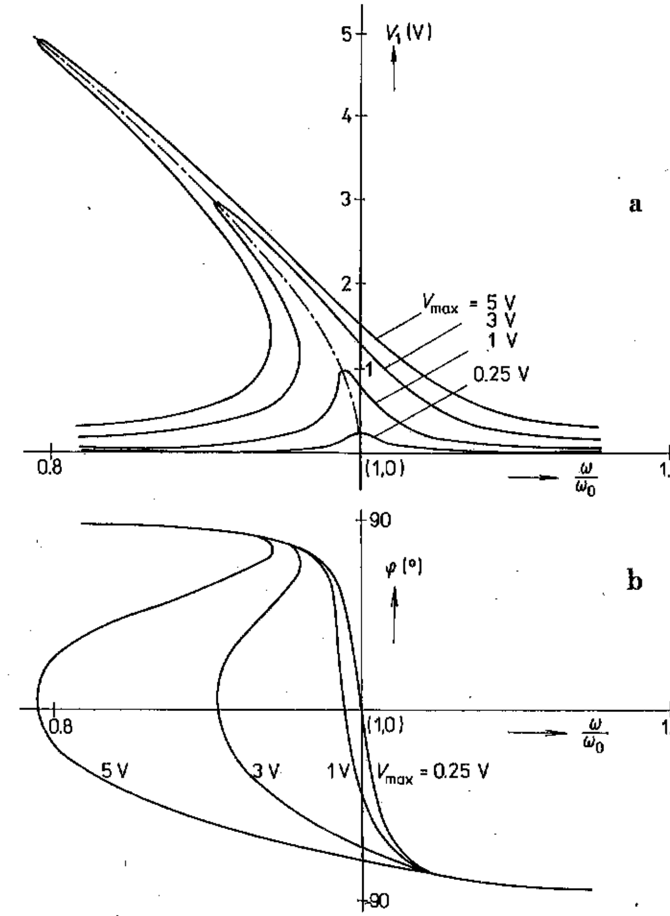


Fig. 229. The set of resonance and phase characteristics of a resonant circuit with a capacitive diode, calculated with the aid of a computer

Hence

$$\omega = \frac{1}{\sqrt{LC_e(V_1)}}\sqrt{1 \pm \sqrt{\frac{1}{Q^2}\left(\frac{V_{max}^2}{V_1^2} - 1\right)}}.$$

Substituting now for $C_e(V_1)$ from Eq. (10.37), we obtain the equation

$$\omega = \frac{\omega_0}{\sqrt{1 + \frac{3}{16}aV_1^2}}\sqrt{1 \pm \sqrt{\frac{1}{Q_2}\left(\frac{V_{max}^2}{V_1^2} - 1\right)}}. \tag{10.44}$$

A family of resonance curves $V_1 = f(\omega/\omega_0)$, calculated according to Eq. (10.44) for several values of $V_{max}$, is given in Fig. 229a. We can see that for small values of $V_{max}$ (e.g. $V_{max} = 0.25$ V) the resonance curves are of the same shape as in the case of the linear resonant circuit (the capacitance scarcely changes with voltage). With increasing $V_{max}$ the curves begin to be deformed, and their maxima tilt toward lower frequencies (because the equivalent capacitance increases).

The equation corresponding to the argument (phase) characteristic is obtained on the basis of Eq. (10.39) rewritten in the form

$$\mathbf{V}_1 = \operatorname{Re} \mathbf{V}_1 + \mathrm{j} \operatorname{Im} \mathbf{V}_1 = I \frac{\omega^2 L^2 R_p}{R_p^2[1 - \omega^2 LC_e(V_1)]^2 + \omega^2 L^2} + \mathrm{j} I \frac{\omega L R_p^2 [1 - \omega^2 LC_e(V_1)]}{R_p^2[1 - \omega^2 LC_e(V_1)]^2 + \omega^2 L^2}. \tag{10.45}$$

Hence

$$\varphi = \arg \mathbf{V}_1 = \arctan \frac{\operatorname{Im} \mathbf{V}_1}{\operatorname{Re} \mathbf{V}_1} = \arctan \frac{R_p}{\omega L}[1 - \omega^2 LC_e(V_1)] = \arctan \frac{R_p}{\omega L}\left[1 - \frac{\omega^2}{\omega_0^2}\left(1 + \frac{3}{16} a V_1^2\right)\right]$$

or

$$\varphi = \arctan Q\left[1 - \frac{\omega^2}{\omega_0^2}\left(1 + \frac{3}{16} a V_1^2\right)\right]. \tag{10.46}$$

Though it is not obvious at first sight from Eq. (10.46), the phase shift $\varphi$ is a function of the amplitude $V_1$ and the amplitude $V_{max} = IR_p$. The dependence on the parameter $V_{max}$ consists in that by Eq. (10.44) the frequency $\omega = f(V_1, V_{max})$. The family of phase characteristics $\varphi(\omega/\omega_0)$ is plotted for several values of $V_{max}$ in Fig. 229b.

J u m p  p h e n o m e n a  a t  n o n - l i n e a r  r e s o n a n c e. Fig. 229a shows the typical, theoretically calculated shape of the resonance characteristics of the damped non-linear resonant circuit. It is obvious that up to a certain magnitude of the voltage amplitude $V_{max}$ (in the given case $V_{max} < 1$ V) the resonance curves are single-valued. For higher voltage amplitudes, the resonance curves are multivalued. Consider the curve shown in Fig. 230 ($V_{max} = 3$ V) and observe the change in the amplitude $V_1$ of the voltage across the capacitor as the frequency increases. It can be seen that with increasing relative angular frequency $\omega/\omega_0 = \alpha$ the amplitude $V_1$ increases continuously at first (from point $A_1$ to points $A_2$, and $A_3$), but at point $A_3$ it jumps to the amplitude given by point $A_4$. With further



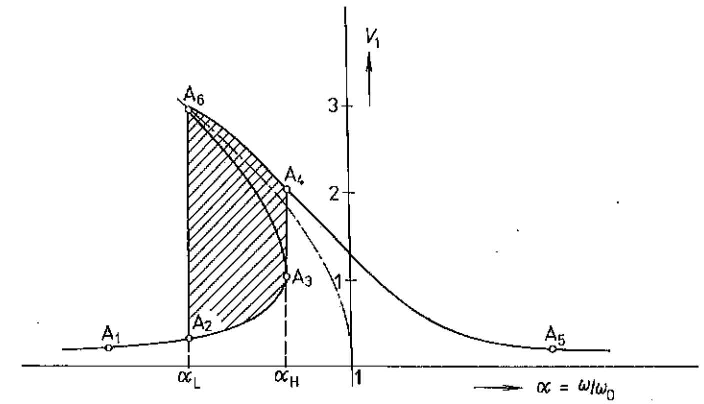


Fig. 230. Jump changes in amplitude at resonance in a resonant circuit with a non-linear capacitor

increases in the relative frequency, the amplitude decreases again continuously (from point $A_4$ to point $A_5$). If, on the contrary, we reduce the relative frequency, the voltage amplitude will increase continuously (points $A_5$, $A_4$, $A_6$). On reaching point $A_6$, it will jump down to the amplitude corresponding to point $A_2$, and with further decreasing $\omega/\omega_0$ it will decrease continuously again. The shaded region is the instability region of the circuit ($\alpha \in \langle \alpha_L, \alpha_H \rangle$), because in this region the resonance characteristic is multivalued. We also come across these jump phenomena in the amplitude when measuring the resonance characteristics of parametric resonant circuits. The theoretically established shape of the characteristic between points $A_3$ and $A_6$ in Fig. 230 cannot be verified experimentally [9], [19], [33].

### 10.2.2 Resonance phenomena in parametric resonant circuits

The parametric resonant circuit is a circuit containing *at least one controlled storage element*. As we know, such elements can store energy or they can deliver energy. Under certain conditions, parametric storage elements have the property of drawing energy at one frequency and supplying it at another frequency. In these elements, there is an energy transposition taking place in the frequency spectrum. This phenomenon is exploited in parametric generators and amplifiers.

In generators (i.e. oscillators) and amplifiers the necessary energy is drawn from a dc voltage (or current) source which must form part of

these circuits. From the point of view of energy, such amplifiers and oscillators can be regarded as converters of dc voltage energy to ac voltage or current energy.

In parametric amplifiers and oscillators, energy is conveyed to the circuit by means of a periodic change in the parameter of a storage element, for which a certain amount of energy is consumed which must be supplied from the source that alters this parameter. Since the parameter of a storage element changes at one frequency and the amplification or excitation of oscillations usually takes place at another frequency, we are concerned here with circuits in which a transposition of energy takes place in the spectrum.

The periodic change in the parameter of the storage element can be realized not only by electrical but also by non-electrical quantities. In that case, of course, mechanical, light, or thermal energy is converted to electrical energy, and the parametric circuit has the function of an energy converter.

The phenomena in parametric circuits are described by parametric equations, i.e. equations with periodically varying coefficients. For their solution we use, on the one hand, some of the methods for the analysis of linear and non-linear circuits (the method of equivalent linearization, the method of slowly changing amplitudes, the phase plane method, or some other method), and on the other hand, some less familiar methods. The latter include in the first place the methods for solving Mathieu's and Hill's differential equations.

Note that in present-day parametric amplifiers and generators, non-linear capacitors and inductors are often used in the function of auto-parametric elements.

Fundamental physical processes in parametric excitation of oscillations. Before examining the conditions of self-excitation of oscillations and steady processes in a parametric resonant circuit, we shall first give the fundamental physical processes taking place in the circuit when there is a periodic change in the parameter of the storage element. We shall explain how energy is exchanged between the circuit and the source altering the parameter of the storage element.

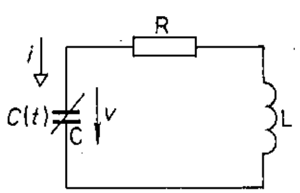


Fig. 231. Parametric resonant circuit



Consider a simple parametric resonant circuit in which the capacitance of the capacitor changes (Fig. 231). Assume that there is a certain initial charge $Q_1$ on the electrodes of the capacitor C. If the capacitance of this capacitor is $C_1$, the voltage across it will be $V_1 = Q_1/C_1$ and the energy stored in its electric field will have the magnitude

$$W_1 = \frac{1}{2} Q_1 V_1. \tag{10.47}$$

With the capacitance decreased from the value $C_1$ to the value $C_2 = C_1 - \Delta C$, this energy will increase to the value

$$W_2 = \frac{1}{2} Q_1 V_2 = \frac{C_1}{C_2} W_1, \tag{10.48}$$

where

$$V_2 = \frac{C_1}{C_2} V_1. \tag{10.49}$$

This is because the charge of the capacitor has not changed with the decrease in its capacitance ($Q_1 = C_1 V_1 = C_2 V_2$). The increment $\Delta W$ of energy in the capacitor is given by the difference of the two voltages

$$\Delta W = W_2 - W_1 = \frac{1}{2} Q_1 (V_2 - V_1) = \frac{1}{2} Q_1 \,\Delta V. \tag{10.50}$$

It follows from Eqs (10.48) and (10.49) that the increase in the voltage across the capacitor and the energy stored in the capacitor will be greater as the change in the capacitance increases. If the capacitance of the capacitor changes, for instance, in the ratio $C_1/C_2 = 5$, the voltage across the capacitor and, consequently, the energy in it will increase five times: $V_2 = 5V_1$ and $W_2 = 5W_1$. This increase in energy is at the expense of the work which the source changing the capacitance of the capacitor must exert against the field strength in the capacitor.

Assume now that the capacitance of a plate capacitor connected in the resonant circuit changes periodically as shown in Fig. 232a, due, say, to the action of a mechanical force which changes stepwise the distance between the plates. The period of the change in the capacitance $T_1 = 2\pi/\omega_1$. At time $t = 0$, let all the energy in the resonant circuit be concentrated in the capacitor, and let its capacitance at this instant change stepwise by a value $\Delta C$. The voltage across the capacitor will then increase, and so will the energy stored in it. Then an energy exchange between the capacitor and the inductor takes place. During a quarter of a period of free oscil-

lations in the circuit all the energy is concentrated in the inductor, and the voltage across the capacitor equals zero. If at this instant we increase the capacitance of the capacitor stepwise to the original value, this will happen without any energy consumption because $W_C = 0$. The energy stored in the magnetic field of the inductor will remain unchanged. Next comes the exchange of energy between the inductor and the capacitor. After the next quarter of a period of free oscillations all the energy of the circuit is again stored in the capacitor. If at this time we reduce stepwise the capacitance of the capacitor, a new supply of energy will be conveyed to the capacitor, and hence also to free oscillations in the resonant circuit. The above process makes it possible to deliver permanently energy from the source changing the capacitance to the resonant circuit. If the energy supplied to the circuit is greater than the energy absorbed in the resistor R, the oscillations in the circuit will increase (Fig. 232b). In the above case, the energy is supplied to the circuit twice per period of oscillation.

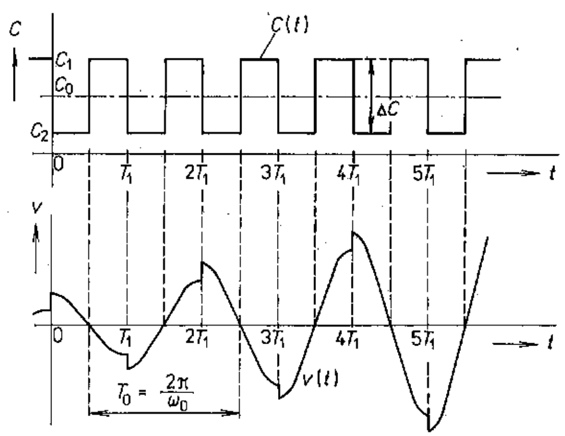


Fig. 232. Illustrating the process of parametric excitation of oscillations

Let us now establish, at least approximately, the condition which must be satisfied if the oscillations in a parametric resonant circuit are to increase. Consider that the current $i(t) = I\cos(\omega_0 t + \pi/2)$ in the circuit is almost harmonic with period $2\pi/\omega_0 = T_0 = 2T_1$. Assume further that the capacitance of the capacitor C always decreases at instants $nT_1$ ($n$ is integer), when the absolute value of the voltage $v$ is maximum, i.e. $|v| = V$. Next, introduce the relative change in capacitance $\Delta C/C_0$, where $\Delta C = C_1 - C_2$ and $C_0 = (C_1 + C_2)/2$.



If the oscillations are to increase, then the energy dissipated in the resistor R during half a period of oscillations,

$$\Delta W_R = \frac{RI^2}{2}\,\frac{T_0}{2}, \tag{10.51}$$

must be less than the increment of energy

$$\Delta W_C = \frac{\Delta C V^2}{2} \tag{10.52}$$

supplied to the circuit periodically every half period. Since $I = \omega_0 C_0 V$, we can formulate for the increase in oscillations in the circuit the condition

$$\Delta W_C > \Delta W_R \quad \text{i.e.} \quad \frac{\Delta C}{C_0} > \omega_0 C_0 R\pi. \tag{10.53}$$

Denoting the quality factor of the resonant circuit $Q = 1/(\omega_0 C_0 R)$, we can simplify this relation still further, namely to the form

$$\frac{\Delta C}{C_0} > \frac{\pi}{Q}. \tag{10.54}$$

We have so far considered the idealized case of parametric excitation of oscillations based on step changes in the capacitance. It can be shown, however, that oscillations in a parametric resonant circuit can increase also in cases when the capacitance $C(t)$ changes in general periodically, namely with the fundamental frequency $\omega \approx 2\omega_0$, where $\omega_0 = 1/\sqrt{C_0 L}$ is the natural frequency of the resonant circuit.

M a t h i e u ' s e q u a t i o n a n d i t s s o l u t i o n. Investigating the conditions of the self-excitation of oscillations and analysing both transient and steady processes in non-linear parametric circuits in general form is very difficult. These processes are described by non-linear differential equations with time-varying coefficients. In the analysis of parametric circuits we therefore proceed in such a way that we examine separately the above conditions and processes; this will yield certain simplifications in the solution.

The process of exciting oscillations in a parametric resonant circuit begins in the same fashion as in the case of common oscillators, namely by the fluctuation of elementary charges in the circuit. At the beginning of the generation, the amplitudes of oscillations are very small, so that their effect on the element parameters need not be considered. Thus the conditions of self-excitation of oscillations can be investigated in the linear

parametric resonant circuit; the processes will be described by linear differential equations with time-varying coefficients. In the examination of steady-state processes, the solution of the non-linear differential equation with time-varying coefficients is simplified, since the amplitudes and the phases of oscillations in the parametric generator do not change.

The conditions of self-excitation of oscillations in the linear parametric circuit of Fig. 231 will be investigated by writing the respective differential equation and finding its solution. We shall assume that the capacitance $C(t)$ varies by the time function

$$C(t) = \frac{C_0}{1 + m\cos(2\omega t)}, \tag{10.55}$$

where $m = C_1/C_0 \ll 1$ is the relative amplitude of the change in the capacitance ($C_1$ is the amplitude of the change and $C_0$ the quiescent value of the capacitance).

The differential equation for the linear parametric resonant circuit can be written in the form:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C(t)}q = 0. \tag{10.56}$$

After substituting for $C(t)$ and rewriting, we obtain

$$\frac{d^2q}{dt^2} + 2\alpha\frac{dq}{dt} + \omega_0^2(1 + m\cos(2\omega t))\,q = 0. \tag{10.57}$$

Here, similarly to the linear resonant circuit with constant parameters, we have

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \omega_0^2 = \frac{1}{LC_0}, \tag{10.58}$$

where $\alpha$ is the damping factor, and $\omega_0$ represents the resonant frequency of the circuit for $m \to 0$. Denoting

$$\tau = \omega t, \quad c = \frac{\omega_0^2}{\omega^2}, \quad 2b = m\frac{\omega_0^2}{\omega^2}, \quad 2d = \frac{2\alpha}{\omega}, \tag{10.59}$$

then

$$\frac{dq}{dt} = \frac{dq}{d\tau}\frac{d\tau}{dt} = \omega\frac{dq}{d\tau}, \tag{10.60}$$

$$\frac{d^2q}{dt^2} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \omega^2\frac{d^2q}{d\tau^2}, \tag{10.61}$$



and Eq. (10.57) can be rewritten in the form

$$\frac{d^2q}{d\tau^2} + 2d\frac{dq}{d\tau} + (c + 2b\cos(2\tau))\,q = 0. \tag{10.62}$$

Substituting

$$q = y\,e^{-d\tau} \tag{10.63}$$

we can eliminate from Eq. (10.62) the term containing the first derivative of the charge. If we twice differentiate Eq. (10.63),

$$\frac{dq}{d\tau} = \frac{dy}{d\tau}e^{-d\tau} - d\,e^{-d\tau}y,$$

$$\frac{d^2q}{d\tau^2} = \frac{d^2y}{d\tau^2}e^{-d\tau} - 2d\,e^{-d\tau}\frac{dy}{d\tau} + d^2e^{-d\tau}y,$$

and substitute these expressions into Eq. (10.62), we obtain Mathieu's equation

$$\frac{d^2y}{d\tau^2} + (a + 2b\cos 2\tau)\,y = 0 \tag{10.64}$$

in which

$$a = c - d^2 = \frac{\omega_0^2}{\omega^2} - d^2. \tag{10.65}$$

Its solution can be expressed by the sum of two linearly independent solutions [4], [33]

$$y = y_1 + y_2 = A_1 e^{\mu\tau}F(\tau) + A_2 e^{-\mu\tau}F(-\tau). \tag{10.66}$$

Here, $A_1$ and $A_2$ are constants which can be found from the initial conditions, $F(\tau)$ and $F(-\tau)$ are periodic functions with period $\pi$ or $2\pi$, and $\mu$ is a coefficient depending on $a$ and $b$, and it can be imaginary or real.

The result (10.66) consists of the sum of two nearly periodic functions whose amplitudes ($A_1e^{\mu\tau}$ and $A_2e^{-\mu\tau}$) increase or are damped if the quantity $\mu$ is real. When investigating the nature of the solution by Eq. (10.66), we are in the first place interested in the problem of stability, i.e. in determining the conditions under which this solution represents oscillations with increasing amplitude. It is obvious that this will happen if the quantity $\mu$ is real, non-zero, positive, or negative:

$$|\mu| > 0. \tag{10.67}$$

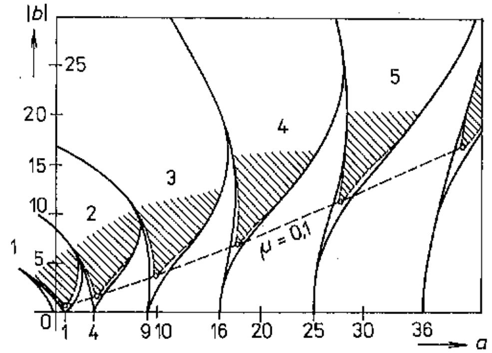


Fig. 233. Graphical representation of instability regions in a plane (the Ince-Strutt chart)

In that case, the amplitude of one of the components of solution (10.66) will increase beyond all bounds. Inequality (10.67) thus expresses the condition of the self-excitation of a parametric circuit which is, of course, lossless.

In Fig. 233 we give the graphical representation of the instability regions in the $a-b$ plane (the Ince-Strutt chart [4]). Inside the instability regions, the coefficient $\mu$ has real values. The region boundaries correspond to $\mu = 0$; in the direction towards the inside of the region, the quantity $|\mu|$ increases. The instability regions (denoted in Fig. 233 by 1 through 5) are bounded by curves which converge on the $a$ axis (i.e. at $b = 0$) at points at which $a = n^2$, where $n = 2\omega_0/\omega_1 = 1, 2, 3, \ldots$. This means that for $a = 1, 4, 9, 16, \ldots$, i.e. for $\omega_1 = 2\omega_0$, $\omega_1 = \omega_0$, $\omega_1 = 2\omega_0/3$, $\omega_1 = \omega_0/2$, etc., the solution of Mathieu's equation is unstable for any arbitrarily small relative amplitude of the capacitance change $m$ (i.e. for $b \to 0$).

If the parameters $a, b$ in Eq. (10.64) correspond to some of the instability regions, self-excited oscillations appear in the parametric resonant circuit. At points which are outside the instability regions, the coefficient $\mu$ assumes a purely imaginary value, which does not result in increasing oscillations. For $b < 0$, the instability regions are shaped by symmetrical reflection about the axis $a$; plotted on the vertical axis in Fig. 233 is the absolute value $|b|$.

It is obvious from Fig. 233 that the most favourable conditions for parametric excitation of oscillations exist in the neighbourhood of point $a = 1$, i.e. for $\omega_1 = 2\omega_0$, when energy is being delivered to the circuit twice per period of free oscillations. For $\omega_1 \neq 2\omega_0$, when the capacitance does not change at the same rate as the free oscillations in the circuit, the value of $b$ must be the greater, the more frequency $\omega_1$ differs from frequency $2\omega_2$ if instability is to be attained.



As has been said earlier, the boundaries of instability regions marked in Fig. 233 by the heavy line hold for $\mu = 0$, i.e. for a lossless resonant circuit in which $\alpha = 0$ and thus also $d = 0$. In an actual resonant circuit with losses there is $\alpha > 0$, and hence also $d > 0$. The coefficient $a$ equals

$$a = c - d^2 = \frac{\omega^2 - \alpha^2}{\omega^2}. \tag{10.68}$$

For a resonant circuit with losses, the solution will by Eqs (10.63) and (10.66) be in the form

$$y = A_1\,\mathrm{e}^{(\mu-d)\tau}\mathrm{F}(\tau) + A_2\,\mathrm{e}^{-(\mu+d)\tau}\mathrm{F}(-\tau). \tag{10.69}$$

The solution will be unstable if $\mu$ is a real quantity, with

$$|\mu| > d. \tag{10.70}$$

The instability regions will be bounded by curves for which $|\mu| = d$. These curves are within the regions $\mu = 0$; they do not touch the $a$ axis, they form a set of tongues, the lowest points of which are the farther from the $a$ axis, the larger $\mu$ is. In Fig. 233, the instability regions for $|\mu| = d = 0.1$ are indicated by shading. The system loses stability most easily when the values of $b$ are minimum, i.e. in conditions corresponding to the minimum of the limit curves.

It can be proved that the same conclusions also hold for the parametric circuit with time-varying inductance.

Calculation of the amplitude of parametric oscillations. In the investigation of the conditions for self-excitation of oscillations in the parametric generator we have so far assumed that all its elements are linear. In such a case the amplitude of oscillations would increase beyond all bounds. In fact, however, one of the storage elements (or both of them) is non-linear, which results in a limitation of the amplitude of the generated harmonic oscillations. If the amplitude of the oscillations in the circuit reaches a certain magnitude, then owing to the non-linear characteristic of the capacitor or inductor the capacitance or inductance in the circuit starts changing (increasing or decreasing), which leads to detuning of the circuit with respect to the frequency of its harmonic oscillations and thus also to deteriorated conditions of parametric excitation. In connection with this we are interested in the amplitude-frequency characteristic of the non-linear parametric circuit.

We shall examine the above characteristics of a resonant circuit containing a controlled non-linear capacitor (Fig. 234), the inductor and the resistor being linear elements. Let the elastance $D = 1/C$ of the

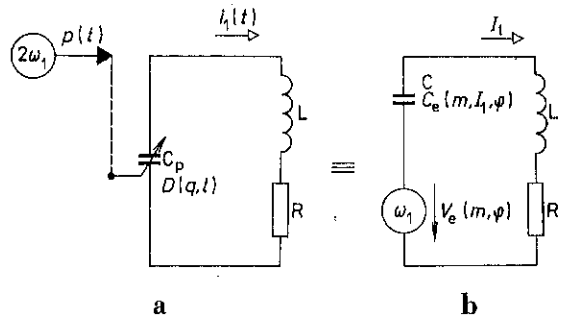


Fig. 234. Models of the resonant circuit with a controlled non-linear capacitor

capacitor vary by the time function

$$D(q, t) = D_0(1 + m \sin(2\omega_1 t)) + aD_0 q^2, \tag{10.71}$$

and let the resonant circuit be tuned at a small amplitude of the voltage across the resonant circuit to the frequency $\omega_0$, which is equal or close to half the frequency $2\omega_1$, i.e. $\omega_0 \approx \omega_1$. The phenomena in the parametric resonant circuit will then be described by the homogeneous non-linear differential equation with periodically varying coefficient

$$\frac{d^2 q}{dt^2} + 2\alpha \frac{dq}{dt} + \omega_0^2(1 + m \sin(2\omega_1 t))\, q + aD_0 q^3 = 0, \tag{10.72}$$

where

$$\alpha = R/(2L), \quad \omega_0^2 = 1/(LC_0), \qquad C_0 = 1/D_0 .$$

An exact analytical solution of this equation is not known, but it can be solved by approximate methods. We shall use the method of equivalent linearization. For the given operational conditions, we shall replace the controlled non-linear capacitor by an equivalent two-terminal element whose impedance is at the given frequency a non-linear function of the parameters of the harmonic components of the signals acting on the capacitor. If we are concerned with a steady process, the impedance of the equivalent two-terminal element is constant.

To be able to determine this impedance, we must know the harmonic components of current and voltage acting on the capacitor. If in the resonant circuit of Fig. 234a subharmonic oscillations of frequency $\omega_1$ are excited, then flowing through the circuit in steady state will be the current

$$i_1(t) = I_1 \cos(\omega_1 t + \varphi), \tag{10.73}$$



so that the charge on the capacitor is

$$q(t) = \int i_1(t)\,dt = \frac{I_1}{\omega_1} \sin(\omega_1 t + \varphi) = Q_1 \sin(\omega_1 t + \varphi). \tag{10.74}$$

The integration constant $Q_0$, i.e. the charge given by the initial conditions, is considered equal to zero.

The waveform of the voltage $v(t)$ across the capacitor will be determined from the relation

$$v = D(q, t)\, q = D_0(1 + m \sin(2\omega_1 t))\, q + aD_0 q^3. \tag{10.75}$$

In this equation, we substitute for $q$ from relation (10.74) and obtain

$$\begin{aligned} v(t) = {} & D_0 Q_1 \sin(\omega_1 t + \varphi) \\ & + \frac{1}{2} m D_0 Q_1 [\cos(\omega_1 t - \varphi) - \cos(3\omega_1 t + \varphi)] \\ & + aD_0 Q_1^3 \left[\frac{3}{4} \sin(\omega_1 t + \varphi) - \frac{1}{4} \sin(3\omega_1 t + 3\varphi)\right]. \end{aligned} \tag{10.76}$$

Since

$$\begin{aligned} \cos(\omega_1 t - \varphi) &= \cos(\omega_1 t + \varphi - 2\varphi) \\ &= \cos(2\varphi) \cos(\omega_1 t + \varphi) + \sin(2\varphi) \sin(\omega_1 t + \varphi), \end{aligned}$$

there will be, further,

$$\begin{aligned} v(t) = {} & \left[\frac{1}{2} m D_0 Q_1 \cos(2\varphi)\right] \cos(\omega_1 t + \varphi) \\ & + \left[D_0 Q_1 \left(1 + \frac{3}{4} a Q_1^2\right) + \frac{1}{2} m D_0 Q_1 \sin(2\varphi)\right] \sin(\omega_1 t + \varphi) \\ & - \frac{1}{2} m D_0 Q_1 \cos(3\omega_1 t + \varphi) - \frac{1}{4} a D_0 Q_1^3 \sin(3\omega_1 t + 3\varphi). \end{aligned} \tag{10.77}$$

It follows from this equation that the harmonic components of the voltage across the capacitor have, in the given case, frequencies equal to 1/2 and 3/2 of the fundamental frequency $2\omega_1$ with which the capacitance of the controlled capacitor changes. If we consider a resonant circuit with a high quality factor ($Q \gg 1$), then only the component of the voltage $v_1(t)$ with subharmonic frequency $\omega_1$, to which the circuit is tuned, will show in the resonant circuit. Since in further calculations we shall work only with the linear equivalent of the resonant circuit under consideration, we shall write the voltage $v_1(t)$ in complex notation:

$$\mathbf{v}_1(t) = \mathbf{V}_1\,e^{j\omega_1 t} = V_1\,e^{j(\omega_1 t+\varphi)}$$
$$= \left\{ \frac{mI_1}{2\omega_1 C_0}\cos(2\varphi) - j\left[\frac{I_1}{\omega_1 C_0}\left(1+\frac{3aI_1^2}{4\omega_1^2}\right)+\frac{mI_1}{2\omega_1 C_0}\sin(2\varphi)\right]\right\} e^{j(\omega_1 t+\varphi)}. \tag{10.78}$$

If we now regard the controlled non-linear capacitor as a two-terminal element across the terminals of which the harmonic voltage $\mathbf{v}(t)$ is acting and through which the current

$$\mathbf{i}_1(t) = \mathbf{I}_1\,e^{j\omega_1 t} = I_1\,e^{j(\omega_1 t+\varphi)} \tag{10.79}$$

is flowing, we can express the equivalent impedance of the capacitor by the relation

$$\mathbf{Z}_e(\omega_1) = \frac{\mathbf{V}_1}{\mathbf{I}_1} = R_e + jX_e = R_e + \frac{1}{j\omega_1 C_e}. \tag{10.80}$$

By calculation we find that

$$R_e = \frac{m}{2\omega_1 C_0}\cos(2\varphi) \tag{10.81}$$

and

$$C_e = \frac{C_0}{1+\dfrac{3aI_1^2}{4\omega_1^2}+\dfrac{m}{2}\sin(2\varphi)}. \tag{10.82}$$

The following conclusions can be drawn from Eqs (10.80) through (10.82): A controlled non-linear capacitor whose elastance changes harmonically with the frequency $2\omega_1$ can under the above conditions be replaced at frequency $\omega_1$ by the equivalent impedance $\mathbf{Z}_e(\omega_1)$, whose parameters are functions of the quantities $m$, $I_1$ and $\varphi$. The active component $R_e$ of the equivalent impedance can be positive or negative; this depends on the initial phase $\varphi$ (i.e. on the phase relations in the resonant circuit). For $\varphi = \pi(k + 1/2)$, where $k$ is an arbitrary integer, the equivalent resistance $R_e$ will be negative and its magnitude will be

$$R_e = -\frac{m}{2\omega_1 C_0}. \tag{10.83}$$

In such a case the capacitor behaves as a source of energy.

With the help of subharmonic components of voltage and current as expressed by Eqs (10.78) and (10.79), we can calculate the active power



delivered at frequency $\omega_1$ by the controlled non-linear capacitor to the resonant circuit:

$$P_C = \frac{1}{2}\mathrm{Re}(\mathbf{V}_1\mathbf{I}_1^*) = \frac{mI_1^2}{4\omega_1 C_0}\cos(2\varphi). \tag{10.84}$$

Since

$$\frac{I_1}{\omega_1} = Q_1 \quad\text{and}\quad \frac{Q_1}{C_0} = V_1, \tag{10.85}$$

we can write the active power in the form

$$P_C = \frac{1}{4}m\,\omega_1 C_0 V_1^2\cos(2\varphi). \tag{10.86}$$

The maximum active power which a controlled non-linear capacitor can deliver at subharmonic frequency $\omega_1$ to the circuit appears for $\cos(2\varphi) = -1$:

$$P_{C\max} = -\frac{1}{4}m\omega_1 C_0 V_1^2 = -\frac{1}{4}\omega_1 C_1 V_1^2. \tag{10.87}$$

Note that this active power $P_C$ must be delivered to the capacitor from outside in the form of the power necessary for the periodic change in the capacitance of the capacitor. As can be seen, the generated power is the greater, the higher the frequency $\omega_1 = \omega_0$, the greater the relative amplitude of the capacitance change $m = C_1/C_0$, and the greater the energy in the capacitor $W_C = C_0V_1^2/2$, where $V_1$ is the amplitude of the voltage across the capacitor.

To investigate the shape of the resonance characteristic at subharmonic frequency $\omega_1$, we shall use the equivalent linearized model of a parametric non-linear resonant circuit as given in Fig. 234b. For this circuit we shall write an equation whose solution will yield the respective resonance characteristics. Following from Fig. 234b, there will be

$$R + \frac{m}{2\omega_1 C_0}\cos(2\varphi) + j\left[\omega_1 L - \frac{1}{\omega_1 C_0}\left(1+\frac{3aI_1^2}{4\omega_1^2}\right) - \frac{m}{2\omega_1 C_0}\sin(2\varphi)\right] = 0. \tag{10.88}$$

In steady state, the real and the imaginary parts of this equation must equal zero. After introducing the dimensionless quantities

$$x = \frac{\omega_1}{\omega_0}, \quad d = \omega_0 C_0 R, \quad A^2 = a\left(\frac{I_1}{\omega_1}\right)^2, \tag{10.89}$$

where $x$ is the relative frequency, $d$ is the damping factor of the circuit at a small amplitude of the voltage across the capacitor ($V_1 \to 0$), and $A$ is the normalized amplitude of oscillations, we obtain a system of two equations in the normalized form:

$$\left.\begin{aligned} &2dx + m\cos(2\varphi) = 0, \\ &x^2 - \left(1 + \frac{3}{4}A^2\right) - \frac{m}{2}\sin(2\varphi) = 0. \end{aligned}\right\} \tag{10.90}$$

Eliminating from this system of equations the phase angle $\varphi$, we obtain the equation

$$\begin{aligned} x^4 - 2\left(1 - \frac{d^2}{2} + \frac{3}{4}A^2\right)x^2 + \left(1 + \frac{3}{4}A^2\right)^2 - \left(\frac{m}{2}\right)^2 \\ = \mathrm{f}(A^2, x^2) = 0. \end{aligned} \tag{10.91}$$

By solving this equation, we obtain the equation of the resonance characteristics in the form

$$x^2 = \left(1 - \frac{d^2}{2} + \frac{3}{4}A^2\right) + \frac{1}{2}\sqrt{m^2 - 4d^2\left(1 + \frac{3}{4}A^2\right) + d^4}. \tag{10.92}$$

The family of resonance characteristics $A = \mathrm{f}(x)$ with the parameter $m$, plotted in accordance with Eq. (10.92), is given in Fig. 235.

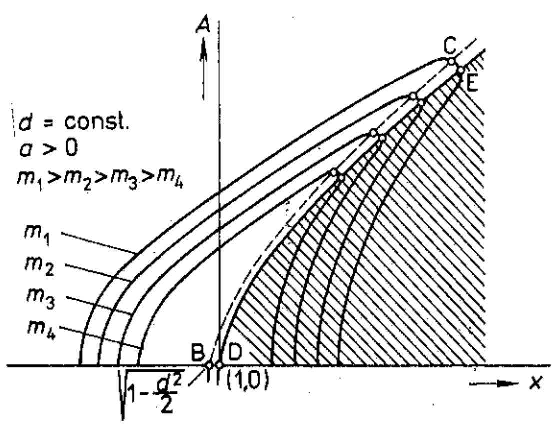


Fig. 235. The set of resonance characteristics of a parametric resonant circuit at subharmonic frequency, plotted on the basis of Eq. (10.92)

From Eq. (10.91), we can derive the equation for the curve BC which represents the locus of the maxima of resonance characteristics (at these points the tangents to the curves are parallel with the abscissa



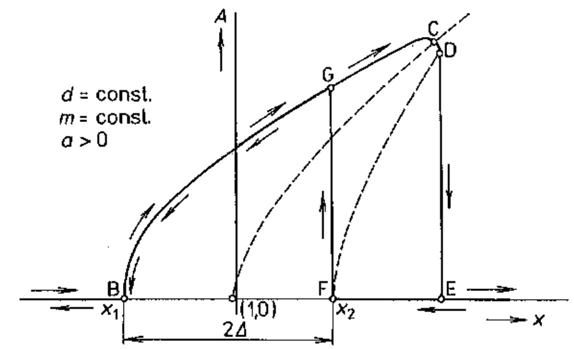


Fig. 236. The measured shape of the resonance characteristic of a parametric resonant circuit at subharmonic frequency

axis, i.e. with the $x$ axis). The equation of the curve will be derived from the condition

$$\frac{\partial \mathrm{f}(A^2, x^2)}{\partial (x^2)} = 0. \tag{10.93}$$

The equation for the curve BC will have the form

$$x^2 - \left(1 - \frac{d^2}{2} + \frac{3}{4}A^2\right) = 0$$

or

$$x^2 - \left(1 - \frac{d^2}{2} + \frac{3}{4}a\frac{I_1^2}{\omega^2}\right) = 0. \tag{10.94}$$

It is obvious that for $a > 0$ the resonance characteristics will be tilted to the right, for $a < 0$ to the left.

The equation for the curve DE, which represents the locus of the points at which the tangents to the resonance curves are vertical, will be found from the condition

$$\frac{\partial \mathrm{f}(A^2, x^2)}{\partial (A^2)} = 0. \tag{10.95}$$

Hence

$$A^2 = \frac{4}{3}(x^2 - 1). \tag{10.96}$$

In the resonance curves of the parametric circuit with particular parameters $m$ and $d$, we are further interested in the bandwidth $2\Delta$ (Fig. 236) in which the subharmonic oscillations are self-excited. It is obvious from Fig. 236 that this bandwidth can be found from the condi-

tion that at the end points B and F, with the abscissas $x_1$ and $x_2$, the amplitude $A$ of the oscillations equals zero. From Eq. (10.92) we thus obtain (neglecting the very small quantity $d^4$) the relation

$$x_{1,2} \approx \sqrt{\left(1 - \frac{d^2}{2}\right) \pm \frac{1}{2}\sqrt{m^2 - 4d^2}}, \tag{10.97}$$

so that

$$2\Delta = x_2 - x_1 \approx \sqrt{\left(1 - \frac{d^2}{2}\right) + \frac{1}{2}\sqrt{m^2 - 4d^2}} - \sqrt{\left(1 - \frac{d^2}{2}\right) - \frac{1}{2}\sqrt{m^2 - 4d^2}}. \tag{10.98}$$

On the assumption that

$$\frac{1}{2}\sqrt{m^2 - 4d^2} \ll 1 - \frac{d^2}{2}, \tag{10.99}$$

the bandwidth $2\Delta$ will be given by the approximate relation

$$2\Delta = x_2 - x_1 \approx \frac{1}{2}\sqrt{m^2 - 4d^2}. \tag{10.100}$$

It can be seen that the bandwidth $2\Delta$ in which parametric oscillations are self-excited depends on the relative capacitance change $m$ and the damping factor $d$ of the resonant circuit. The greater the relative capacitance change $m$ and the smaller the damping factor $d$, the greater the bandwidth will be in which parametric oscillations can be excited.

For a given resonant circuit with damping factor $d$ we can, from Eq. (10.100), also find the minimum, so called critical value $m_c$, at which parametric oscillations can still be excited:

$$m_c = 2d = \frac{2}{Q}. \tag{10.101}$$

We shall still examine the effect of the detuning of a resonant circuit on the existence of parametric oscillations. Denoting the detuning of a resonant circuit by $\Delta\omega$, we can express the relative bandwidth as follows:

$$x_2 - x_1 = \frac{\omega_1 + \Delta\omega}{\omega_0} - \frac{\omega_1 - \Delta\omega}{\omega_0} = \frac{2\Delta\omega}{\omega_0}. \tag{10.102}$$

Then

$$m_c = 2\sqrt{d^2 + \left(\frac{2\Delta\omega}{\omega_0}\right)^2}$$



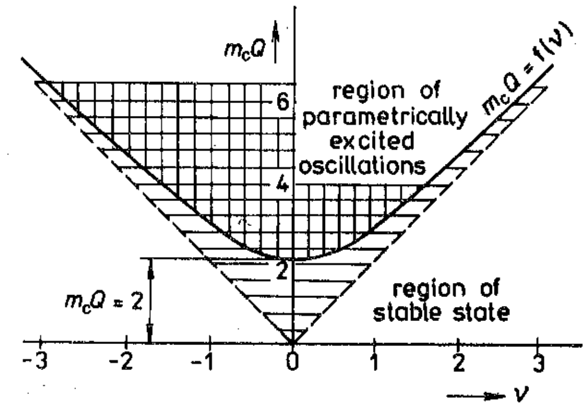


Fig. 237. Graphical representation of the region of self-excitation of parametric oscillations

or

$$\frac{m_c}{d} = m_cQ = 2\sqrt{1 + \left(\frac{2\Delta\omega}{\omega_0}Q\right)^2}, \tag{10.103}$$

where $Q$ is the quality factor of the resonant circuit at $V_1 \to 0$.

Fig. 237 illustrates the dependence $m_cQ = f(v)$, where $v = 2\Delta\omega Q/\omega_0$ (the heavy solid line). The double-shaded region represents the region of the existence of parametric oscillations. The dashed lines are the asymptotes of the curve $m_cQ = \sqrt{1 + v^2}$, and they represent the boundaries of the instability region of a parametrically excited circuit without losses [33].

It is obvious from Fig. 237 that in the case of a detuned resonant circuit it is necessary to increase the minimum value of the relative capacitance change $m_c$, so that parametric oscillations may be excited in the circuit.

The resonance characteristics given in Fig. 235 are theoretically calculated curves. The measured resonance characteristics differ from the theoretical ones: in the instability region, where the resonance characteristic is multivalued, the amplitude of oscillations changes stepwise (at points D and F in Fig. 236).